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General regular charged space-times in teleparallel equivalent of general relativity

G.G.L. Nashed[a]

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

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Abstract. Using a non-linear version of electrodynamics coupled to the teleparallel equivalent of general relativity (TEGR), we obtain new regular exact solutions. The non-linear theory reduces to the Maxwell one in the weak limit with the tetrad fields corresponding to a charged space-time. We then apply the energy-momentum tensor of the gravitational field, established in the Hamiltonian structure of the TEGR, to the solutions obtained.

1 Introduction

It is usually asserted in the literature that the principle of equivalence prevents the gravitational energy from being localizable. However, an expression for the gravitational field energy has been pursued since the early days of general relativity (GR). A considerable amount of efforts have been devoted to finding a viable expression other than the pseudotensor one (an expression for the quasi-local energy, i.e., the energy associated to a closed spacelike two-surface, in the context of the Hilbert–Einstein action integral, has emerged as a tentative description of the gravitational energy [1–5]). The search for a consistent expression for the gravitational energy is undoubtedly a long-standing problem in GR. The argument based on the principle of equivalence regarding the non-localizability of the gravitational energy is controversial and has not generally been accepted [6, 7]. The principle of equivalence does not preclude the existence of scalar densities on the space-time manifold, constructed out of tetrad fields, that may eventually yield the correct description of the energy properties of the gravitational field. Such densities may be given in terms of the torsion tensor, which cannot be made to vanish at a point by a coordinate transformation. Møller was the first one to notice that the tetrad field description of the gravitational field allows for a more satisfactory treatment of the gravitational energy-momentum [8–10].

For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well known that there exists no covariant, non-trivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus, it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than that they are an intrinsic drawback of the theory [6, 11]. Møller has shown that the problem of the energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian space-time [12]. In a series of papers, [12–15] he was able to obtain a general expression for a satisfactory energy-momentum complex in teleparallel space-time.

At present, teleparallel theory seems to be popular again, and there is a trend of analyzing the basic solutions of general relativity with teleparallel theory and comparing the results. The teleparallel equivalent of general relativity (TEGR) is a viable alternative geometrical description of Einstein's general relativity written in terms of the tetrad field [16]. It continues to be the object of thorough investigations [6–37]. In the framework of the TEGR it has been possible to address the long-standing problem of defining the energy, momentum and angular momentum of the gravitational field [38, 39]. The tetrad field seems to be a suitable field quantity to address this problem, because it yields the gravitational field and at the same time establishes a class of reference frames in space-time [40]. Moreover, there are simple and clear indications that the gravitational energy-momentum defined in the context of the TEGR provides a unified picture of the concept of mass-energy in special and general relativity.

Teleparallel theories of gravity are considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [29, 42–49] or metric affine gravity [49]. Physics relevant to geometry may be related to the teleparallel description of gravity [50, 51]. The teleparallel approach is used for the proof that the gravitational energy is positive [52]. The relation between the

[a] e-mail: nashed652001@yahoo.com,
nashed@asunet.shams.edu.eg

spinor Lagrangian and teleparallel theory is established in [53]. It has been shown that the TEGR is not consistent in the presence of minimally coupled spinning matter [54]. The consistence of the coupling of the Dirac fields to the TEGR is also treated in [55]. However, it has been shown that this demonstration is not correct [56]. Also the general teleparallel gravity model within the framework of metric affine gravity theory has been studied in [57].

In fact, the first attempt to construct a theory of the gravitational field in terms of a set of four linearly independent vector fields in the Weitzenböck geometry is due to Einstein [58–60]. The general form of the tetrad field, ${e_a}^\mu$, having spherical symmetry was given by Robertson [61]. In Cartesian form it can be written as[1]

$$\begin{aligned} {e_{(0)}}^0 &= A\,, \quad {e_{(a)}}^0 = Cx^a\,, \\ {e_{(0)}}^\alpha &= Dx^\alpha \quad {e_a}^\alpha = \delta_a^\alpha B + Fx^a x^\alpha + \epsilon_{a\alpha\beta} S x^\beta\,, \end{aligned} \tag{1}$$

where A, C, D, B, F, and S are functions of t and $r = (x^\alpha x^\alpha)^{1/2}$. We consider an asymptotically flat space-time in this paper, and we impose the boundary condition that for $r \to \infty$ the tetrad field of (1) approaches the tetrad of Minkowski space-time, $({e_a}^\mu) = \mathrm{diag}(1, {\delta_a}^\alpha)$.

It is the aim of the present paper to find asymptotically flat solutions with spherical symmetry, which is different from the Schwarzschild and Reissner–Nordström solutions in the TEGR. This can be achieved by inducing the TEGR geometry in non-linear electrodynamics. Applying this philosophy, we obtain two exact solutions. One of these contains an arbitrary function $\Psi(R,t)$, while the other one contains an arbitrary parameter η.

In Sect. 2, we briefly review the TEGR theory of gravitation. We study the general solution without the S-term (see (1)), where the remaining unknown functions are allowed to depend on R and t, and a solution with an arbitrary function $\Psi(R,t)$ is obtained. We compare this solution with that obtained before in [62]. Also in Sect. 2, we study the tetrad field with a non-vanishing S-term (see (1)) from which a solution with one parameter η is obtained. Both solutions give the same metric, which is *a regular charged static spherically symmetric black hole*. Calculations of the energy content of a sphere of radius R using the regularized expression of the gravitational energy-momentum are given in Sect. 3. The final section is devoted to a discussion and our conclusions.

2 The TEGR theory and regular solutions

In a space-time with absolute parallelism the parallel vector fields ${e_a}^\mu$ define the non-symmetric affine connection

$$\Gamma^\lambda_{\mu\nu} \stackrel{\text{def.}}{=} {e_a}^\lambda {e^a}_{\mu,\nu}\,, \tag{2}$$

[1] Space-time indices $\mu, \nu, \cdots$ and SO(3,1) indices a, b, $\cdots$ run from 0 to 3. Time and space indices are assigned to $\mu = 0, i$, and $a = (0), (i)$.

where $e_{a\mu,\nu} = \partial_\nu e_{a\mu}$. The curvature tensor defined by ${\Gamma^\lambda}_{\mu\nu}$, given in (1), is identically vanishing, however. The metric tensor $g_{\mu\nu}$ is given by[2]

$$g_{\mu\nu} = \eta_{ab} {e^a}_\mu {e^b}_\nu\,, \tag{3}$$

with the Minkowski metric $\eta_{a\,b} = \mathrm{diag}(-1, +1, +1, +1)$.

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[3] [6, 40]

$$\begin{aligned} \mathcal{L}_G = eL_G &= -\frac{e}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bzc}}{2} - T^a T_a\right) - L_m \\ &= -\frac{e}{16\pi}\Sigma^{abc}T_{abc} - L_m\,, \end{aligned} \tag{4}$$

where $e = \det({e^a}_\mu)$. The tensor Σ^{abc} is defined by

$$\Sigma^{abc} \stackrel{\text{def.}}{=} \frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) + \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right)\,. \tag{5}$$

T^{abc} and T^a are the torsion tensor, and the basic vector field is defined by

$${T^a}_{\mu\nu} \stackrel{\text{def.}}{=} {e^a}_\lambda {T^\lambda}_{\mu\nu} = \partial_\mu {e^a}_\nu - \partial_\nu {e^a}_\mu\,, \quad T^a \stackrel{\text{def.}}{=} {{T^b}_b}^a\,. \tag{6}$$

It can be shown that the quadratic combination $\Sigma^{abc}T^{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term [11]. L_m represents the Lagrangian density for the matter fields.

The non-linear electrodynamics Lagrangian has the form [70]

$$\mathcal{H}_{\text{n.e.m.}} \stackrel{\text{def.}}{=} -\frac{1}{4}P_{\mu\nu}P^{\mu\rho}\,, \tag{7}$$

with $P_{\mu\nu} = \mathcal{L}(\mathcal{F})_F F_{\mu\nu}$, $\mathcal{L}(\mathcal{F})_F = \frac{\partial\mathcal{L}(\mathcal{F})}{\partial\mathcal{F}}$, $\mathcal{L}(\mathcal{F}) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}$ being the antisymmetric tensor given by[4] $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the vector potential.

The gravitational and non-linear electromagnetic field equations for the system described by $\mathcal{L}_\mathcal{G} + \mathcal{H}_{\text{n.e.m.}}$ have the following form [6, 40, 70]:

$$\begin{aligned} & e_{a\lambda}e_{b\mu}\partial_\nu\left(e\Sigma^{b\lambda\nu}\right) - e\left({\Sigma^{b\nu}}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right) \\ & \quad = \frac{1}{2}\kappa e T_{a\mu}\,, \\ & \partial_\nu\left(\sqrt{-g}P^{\mu\nu}\right) = 0\,, \end{aligned} \tag{8}$$

where $\frac{\delta L_m}{\delta e^{a\mu}} \equiv eT_{a\mu}$. It is possible to prove by explicit calculations that the left hand side of the first equation of (8) is exactly given by $\frac{e}{2}\left\{R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\right\}$ [6]. The energy-momentum tensor $T^{\mu\nu}$ is defined by

$$T^{\mu\nu} \stackrel{\text{def.}}{=} 2\left(\mathcal{H}_P {P^\mu}_\lambda P^{\nu\lambda} - \delta^{\mu\nu}\left\{2P\mathcal{H}_P - \mathcal{H}\right\}\right)\,, \tag{9}$$

[2] Latin indices are raised and lowered with η_{ab} and η^{ab}.

[3] Throughout this paper we use the relativistic units $c = G = 1$ and $\kappa = 8\pi$.

[4] Heaviside–Lorentz rationalized units will be used.

where $P = (1/4)(P_{\mu\nu}P^{\mu\nu})$ and $\mathcal{H}_P = \frac{\partial \mathcal{H}(\mathcal{P})}{\partial P}$.

Now we are going to find regular solutions in the TEGR theory. We will discuss two cases separately: one with $S=0$ and the other with $S \neq 0$; see (1).

2.1 The case without the S-term

In this case the solution of the field (8) is the regular charged black hole [62]. Therefore, the solution of (1) with $S=0$ can be obtained from the diagonal tetrad of the regular charged black hole metric by a *local Lorentz transformation* that keeps spherical symmetry. Namely, we see that

$$e_a{}^{\mu} = \Lambda_a{}^l e_l{}^{(0)\mu} \tag{10}$$

is the most general, regular charged black hole solution without the S-term. Here $e_l{}^{(0)\mu}$ is the diagonal tetrad [62] and $\Lambda_a{}^l$ is the local Lorentz transformation given in [63].

The explicit form of the tetrad field, $e_a{}^{\mu}$, is then given by

$$(e_a^{\mu}) = \begin{pmatrix} \frac{L}{X} & \Psi X & 0 & 0 \\ \frac{\Psi \sin\theta\cos\phi}{X} & LX\sin\theta\cos\phi & \frac{\cos\theta\cos\phi}{R} & -\frac{\sin\phi}{R\sin\theta} \\ \frac{\Psi\sin\theta\sin\phi}{X} & LX\sin\theta\sin\phi & \frac{\cos\theta\sin\phi}{R} & \frac{\cos\phi}{R\sin\theta} \\ \frac{\Psi\cos\theta}{X} & LX\cos\theta & -\frac{\sin\theta}{R} & 0 \end{pmatrix}, \tag{11}$$

where Ψ is an arbitrary function of t and R;

$$L = \sqrt{\Psi^2+1}, \quad X = \left\{1 - \frac{2m}{R}\left[1-\tanh\left(\frac{q^2}{2mR}\right)\right]\right\}^{1/2}, \tag{12}$$

and $R = r/B$ is the Schwarzschild radius. The ansatz of the antisymmetric field $P_{\mu\nu}$, the non-linear electrodynamics source, $\mathcal{H}$, and the energy-momentum tensor, $T_\mu{}^\nu$, used to derive this solution have the form

$$\mathbf{P} = \frac{q}{r^2}\,\mathrm{d}t\wedge\mathrm{d}r, \quad \mathcal{H} = -\frac{q^2}{2r^4}\mathrm{sech}^2\left(\frac{q^2}{2mr}\right),$$
$$T_0{}^0 = T_1{}^1 = \frac{4q^2 \mathrm{e}^{(q^2/mr)}}{8\pi r^4\left(1+\mathrm{e}^{(q^2/mr)}\right)^2},$$
$$T_2{}^2 = T_3{}^3 = \frac{2q^2\mathrm{e}^{(q^2/mr)}\left(q^2(\mathrm{e}^{(q^2/mr)}-1) - 2mr(1+\mathrm{e}^{(q^2/mr)})\right)}{8\pi m r^5 (1+\mathrm{e}^{(q^2/mr)})^3}. \tag{13}$$

The metric associated with the tetrad field given by (11) is by definition given by the regular charged black hole, which has the form

$$\mathrm{d}s^2 = X_1\,\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{X_1} + R^2\,\mathrm{d}\Omega^2, \tag{14}$$

where $X_1 = X^2$ and X are defined in (12).

Now let us compare the solution given by (11) with that given before: we obtained a solution with an arbitrary function $\mathcal{B}$ for the tetrad (1) with three unknown functions in the spherical polar coordinates [62]. The tetrad field of that solution can be obtained from (11) if the arbitrary function Ψ is chosen as

$$\Psi = \frac{\left\{R^2\mathcal{B}'^2 - 2R\mathcal{B}' + \frac{2m}{R}\left[1-\tanh\left(\frac{q^2}{2mr}\right)\right]\right\}^{1/2}}{X}. \tag{15}$$

2.2 The case with non-vanishing S-term

We start with the tetrad field of (1) with the six unknown functions of t and r. In order to study the condition that the field equations (8) are satisfied, it is convenient to start from the general expression for the covariant components of the tetrad field:

$$e^{(0)}{}_0 = \check{A}, \quad e^{(a)}{}_0 = \check{D}x^a, \quad e^{(0)}{}_\alpha = \check{C}x^\alpha,$$
$$e^{(a)}{}_\alpha = \delta^a{}_\alpha \check{B} + \check{F}x^a x_\alpha + \epsilon^a{}_{\alpha\beta}\check{S}x^\beta, \tag{16}$$

where the six unknown functions, $\check{A}$, $\check{C}$, $\check{D}$, $\check{B}$, $\check{F}$ and $\check{S}$ are connected with the six unknown functions of (1). We can assume without loss of generality that the two functions $\check{D}$ and $\check{F}$ are vanishing by making use of the freedom to redefine t and r [50]. We then transform the tetrad field (16) to the spherical polar coordinates (r,θ,ϕ,t), so that we obtain the form given in [63]. Then the condition that the field equations (8) be satisfied is [64]

$$3\check{B}\check{S} + r(\check{B}\check{S}' - \check{B}'\check{S}) = 0, \quad 2\check{C}\check{S} + (\dot{\check{S}}\check{B} - \check{S}\dot{\check{B}}) = 0, \tag{17}$$

with $\check{S}' = \mathrm{d}\check{S}/\mathrm{d}r$ and $\dot{\check{S}} = \mathrm{d}\check{S}/\mathrm{d}t$. Equation (17) can be solved to give

$$\check{C} = 0, \quad \check{S} = \frac{\eta}{r^3}\check{B}, \tag{18}$$

where η is a constant with the dimension of $(length)^2$. The line element then is given by the following expression:

$$\mathrm{d}s^2 = -\check{A}^2\,\mathrm{d}t^2 + \check{B}^2\,\mathrm{d}r^2 + r^2\check{B}^2\left(1+\frac{\eta^2}{r^4}\right)\mathrm{d}^2\Omega. \tag{19}$$

The symmetric part of the field equations now coincides with the Einstein field equations written in terms of the tetrad fields. Therefore, the metric tensor must be the regular charged solution when the Schwarzschild radial coordinate R is used. We defined the new radial coordinate to have the form

$$R = r\check{B}\sqrt{1+\frac{\eta^2}{r^4}} \tag{20}$$

and require that the line element written in the coordinates (t,R,θ,ϕ) coincides with the regular charged metric [70]. Then we have

$$\check{A}(r) = X(R), \quad \frac{\mathrm{d}R}{\mathrm{d}r} = \check{B}(r)X(R). \tag{21}$$

Eliminating $\check{B}$ from (20) and the second equation of (21), we obtain a differential equation for $R(r)$, which can easily be solved to give

$$r^2 = |\eta| \sinh Y(R) , \quad Y(R) = \mathrm{e}^{\int \frac{\mathrm{d}R}{RX}} , \tag{22}$$

where the additive integration constant is fixed in the last equation by requiring the asymptotic condition $r/R \to 1$ as $R \to \infty$. Finally we obtain

$$r\check{B}(r) = R \tanh Y(R) , \quad r^2 \check{S} = \frac{\eta}{r^2} \left(r\check{B}\right) = \frac{\eta}{|\eta|} \frac{R}{\cosh Y(R)} . \tag{23}$$

Now it is straightforward to obtain the covariant components of the tetrad field, ${e^a}_\mu$, with the non-vanishing S-term for the regular charged solution in the coordinate system (t, x^α): the non-vanishing components are given by

$$\begin{aligned} {e^{(0)}}_0 &= X , \\ {e^{(a)}}_\alpha &= \tanh Y {\delta^a}_\alpha + \left(\frac{1}{X} - \tanh Y\right) \frac{x^a x_\alpha}{R^2} \\ &\quad + \left(\frac{\eta}{|\eta|} \frac{1}{\cosh Y}\right) {\epsilon^a}_{\alpha\beta} \frac{x^\beta}{R} . \end{aligned} \tag{24}$$

The solution (24) asymptotically behaves like the solution (20) obtained before in [65], and when $q = 0$ and m is replaced by $m(1 - \mathrm{e}^{-R^3/r1^3})$, it is reduced to the solution (29) obtained in [63].

Finally, we notice that if the constant η is set equal to zero the tetrad field (24) reduces to the matrix inverse of the solution (11) when the arbitrary function $\Psi = 0$.

3 Regularized expression for the gravitational energy-momentum

Multiplication of the first equation of (8) by the appropriate inverse tetrad fields yields the form [6, 40]

$$\begin{aligned} \partial_\nu \left(-e\Sigma^{a\lambda\nu}\right) = &-\frac{ee^{a\mu}}{4} \left(4\Sigma^{b\lambda\nu} T_{b\nu\mu} - {\delta^\lambda}_\mu \Sigma^{bdc} T_{bcd}\right) \\ &- 4\pi {e^a}_\mu T^{\lambda\mu} . \end{aligned} \tag{25}$$

By restricting the space-time index λ, making it assume only spatial values, (25) takes the form [6]

$$\begin{aligned} &\partial_0 \left(e\Sigma^{a0j}\right) + \partial_k \left(e\Sigma^{akj}\right) \\ &= -\frac{ee^{a\mu}}{4} \left(4\Sigma^{bcj} T_{bc\mu} - {\delta^j}_\mu \Sigma^{bcd} T_{bcd}\right) - 4\pi e {e^a}_\mu T^{j\mu} . \end{aligned} \tag{26}$$

Note that the last two indices of Σ^{abc} and T^{abc} are antisymmetric. Taking the divergence of (26) with respect to j yields

$$\begin{aligned} &-\partial_0 \partial_j \left(-\frac{1}{4\pi} e\Sigma^{a0j}\right) = \\ &-\frac{1}{16\pi} \partial_j \left[ee^{a\mu} \left(4\Sigma^{bcj} T_{bc\mu} - {\delta^j}_\mu \Sigma^{bcd} T_{bcd}\right) - \partial_j (e {e^a}_\mu T^{j\mu})\right] . \end{aligned} \tag{27}$$

In the Hamiltonian formulation of the TEGR [28, 66] the momentum canonically conjugate to the tetrad components e_{aj} is given by

$$\Pi^{aj} = -\frac{1}{4\pi} e\Sigma^{a0j} ,$$

and the gravitational energy-momentum P^a contained within a volume V of a three-dimensional spacelike hypersurface is defined by [6]

$$P^a = -\int_V \mathrm{d}^3x \partial_j \Pi^{aj} . \tag{28}$$

If no condition is imposed on the tetrad field, P^a transforms as a vector under the global SO(3,1) group. It describes the gravitational energy-momentum with respect to observers adapted to ${e_a}^\mu$. These observers are characterized by the velocity field $u^\mu = {e_{(0)}}^\mu$ and by the acceleration f^μ

$$f^\mu = \frac{Du^\mu}{\mathrm{d}s} = \frac{D{e_{(0)}}^\mu}{\mathrm{d}s} = u^a \nabla_a {e_{(0)}}^\mu .$$

Let us assume that the space-time is asymptotically flat. The total gravitational energy-momentum is given by

$$P^a = -\oint_{S\to\infty} \mathrm{d}S_k \Pi^{ak} . \tag{29}$$

The field quantities are evaluated on a surface S in the limit $r \to \infty$.

An important property of the tetrad fields that satisfy the condition

$$e_{a\mu} \cong \eta_{a\mu} + (1/2) {h_a}^\mu (1/r) \tag{30}$$

is that in the flat space-time limit ${e_a}^\mu(t, x, y, z) = {\delta_a}^\mu$, and therefore the torsion tensor is vanishing, i.e., ${T^\lambda}_{\mu\nu} = 0$. Hence for the flat space-time it is normal to consider a set of tetrad fields such that ${T^\lambda}_{\mu\nu} = 0$ *in any coordinate system*. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (30). Moreover, for such tetrad fields the torsion tensor obeys ${T^\lambda}_{\mu\nu} \neq 0$ for the flat space-time [6, 40]. It might be argued, therefore, that the expression for the gravitational energy-momentum (29) is restricted to a particular class of tetrad fields, namely, to the class of frames in which ${T^\lambda}_{\mu\nu} = 0$ if ${e_a}^\mu$ represents the flat space-time tetrad field [40]. To explain this, let us calculate the flat space-time of the tetrad field given by (11) (by putting the physical qua ntities m and q equal to zero), to find

$$({E_a}^\mu) = \begin{pmatrix} L & \Psi & 0 & 0 \\ \Psi \sin\theta \cos\phi & L \sin\theta \cos\phi & \frac{\cos\theta \cos\phi}{R} & \frac{-\sin\phi}{R \sin\theta} \\ \Psi \sin\theta \sin\phi & L \sin\theta \sin\phi & \frac{\cos\theta \sin\phi}{R} & \frac{\cos\phi}{R \sin\theta} \\ \Psi \cos\theta & L \cos\theta & \frac{-\sin\theta}{R} & 0 \end{pmatrix} , \tag{31}$$

where L is defined in (12). Using (6), the expression (31) yields the following non-vanishing torsion components:

$$\begin{aligned}
&T_{121}=\cos\theta\cos\phi(1-L)\,, && T_{131}=\sin\theta\sin\phi(L-1)\,,\\
&T_{141}=\frac{\partial\Psi}{\partial R}\sin\theta\cos\phi\,, && T_{124}=\Psi\cos\theta\cos\phi\,,\\
&T_{134}=-\Psi\sin\theta\sin\phi\,, && T_{212}=\cos\theta\sin\phi(L-1)\,,\\
&T_{213}=\sin\theta\cos\phi(L-1)\,, && T_{214}=\frac{\partial\Psi}{\partial R}\sin\theta\sin\phi\,,\\
&T_{224}=\Psi\cos\theta\sin\phi\,, && T_{234}=\Psi\sin\theta\cos\phi\,,\\
&T_{312}=\sin\theta(1-L)\,, && T_{314}=\frac{\partial\Psi}{\partial R}\cos\theta\,,\\
&T_{324}=-\Psi\sin\theta\,, && T_{414}=\frac{\Psi\frac{\partial\Psi}{\partial R}}{L}\,.
\end{aligned}\qquad(32)$$

The tetrad field (31) when written in Cartesian coordinates will have the form

$$({E_a}^\mu(t,x,y,z))=\begin{pmatrix} L & \frac{x^\alpha\Psi}{R}\\ \frac{x^a\Psi}{R} & {\delta_a}^\alpha+\frac{x^ax^\alpha(L-1)}{r^2}\end{pmatrix}\,.\qquad(33)$$

In view of the geometric structure of (33), we see that (11) does not display the asymptotic behavior required by (30). Moreover, in general the tetrad field given by (33) is adapted to accelerated observers [6,40,66]. To explain this, let us consider a boost in the x-direction of (33). We find

$$\begin{aligned}&({E_a}^\mu(t,x^\alpha))\\
&=\begin{pmatrix} \gamma L & -v\gamma\frac{x\Psi}{R} & \frac{y\Psi}{R} & \frac{z\Psi}{R}\\
-v\gamma\frac{x\Psi}{R} & \gamma\left(1+\frac{x^2(L-1)}{r^2}\right) & \frac{xy(L-1)}{r^2} & \frac{xz(L-1)}{r^2}\\
\frac{y\Psi}{R} & \frac{xy(L-1)}{r^2} & 1+\frac{y^2(L-1)}{r^2} & \frac{yz(L-1)}{r^2}\\
\frac{z\Psi}{R} & \frac{xz(L-1)}{r^2} & \frac{yz(L-1)}{r^2} & 1+\frac{z^2(L-1)}{r^2}\end{pmatrix},\end{aligned}\qquad(34)$$

where v is the speed of the observer and $\gamma=\sqrt{1-v^2}$. It can be shown that along an observer's trajectory whose velocity is determined by

$$u^u=E^\mu_{(0)}=\left(\gamma\sqrt{\Psi^2+1},-v\gamma\frac{x\Psi}{R},\frac{y\Psi}{R},\frac{z\Psi}{R}\right)\,,$$

and the quantities

$${\phi_{(j)}}^{(k)}=u^i\left({E^{(k)}}_m\partial_i{E_{(j)}}^m\right)\,,\qquad(35)$$

constructed from (34) are non-vanishing. This fact indicates that along the observer's path the spatial axis ${E_{(i)}}^\mu$ rotates [6,40,66]. In spite of the above problems discussed for the tetrad field of , it yields a satisfactory value for the total gravitational energy-momentum, as we will discuss.

Maluf et al. [40] discussed the above problems for the Kerr space-time and constructed a regularized expression for the gravitational energy-momentum in the framework of TEGR. They checked this expression for the diagonal tetrad field of Kerr space-time that suffers from the above problems and obtain very satisfactory results [40].

In (28) and (29) it is implicitly assumed that the reference space is determined by a set of tetrad fields ${e_a}^\mu$ for flat space-time such that the condition ${T^a}_{\mu\nu}=0$ is satisfied. However, in general there exist flat space-time tetrad fields for which ${T^a}_{\mu\nu}\neq 0$. In this case (28) may be generalized [40,66] by adding a suitable reference space subtraction term, exactly like in the Brown–York formalism [67–69].

We will denote ${T^a}_{\mu\nu}(E)=\partial_\mu {E^a}_\nu-\partial_\nu {E^a}_\mu$ and $\Pi^{aj}(E)$ as the expression of Π^{aj} constructed out of the flat tetrad field ${E_a}^\mu$. The *regularized form* of the gravitational energy-momentum P^a is defined by [40,66]

$$P^a\overset{\text{def.}}{=}-\int_V \mathrm{d}^3x\partial_k\left[\Pi^{ak}(e)-\Pi^{ak}(E)\right]\,.\qquad(36)$$

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by the tetrad fields ${E_a}^\mu$ obtained from ${e_a}^\mu$ by requiring the vanishing of the physical parameters like mass, angular momentum, etc. Assuming that the space-time is asymptotically flat, then (36) may have the form [40,66]

$$P^a=-\oint_{S\to\infty}\mathrm{d}S_k\left[\Pi^{ak}(e)-\Pi^{ak}(E)\right]\,,\qquad(37)$$

where the surface S is established at spacelike infinity. Equation (37) transforms as a vector under the global SO(3,1) group. Now we are in a position to prove that the tetrad field (11) yields a satisfactory value for the total gravitational energy-momentum.

We will integrate (37) over a surface of constant radius, $x^1=r$, and require $r\to\infty$. Therefore, the index k in (37) takes the value $k=1$. We need to calculate the quantity

$$\Sigma^{(0)01}={e^{(0)}}_0\Sigma^{001}=\frac{1}{2}{e^{(0)}}_0(T^{001}-g^{00}T^1)\,.$$

Evaluating the above equation using the tetrad field of (11), we find

$$\begin{aligned}-\Pi^{(0)1}(e)&=\frac{1}{4\pi}e\Sigma^{(0)01}=-\frac{1}{8\pi}R\sin\theta\\
&\times\left(\left\{\sqrt{1-\frac{2m}{R}\left[1-\tanh\left(\frac{q^2}{2mR}\right)\right]}\right\}L(L+1)\right)\,,\end{aligned}\qquad(38)$$

and the expression of $\Pi^{(0)1}(E)$ is given by

$$\Pi^{(0)1}(E)=\frac{1}{8\pi}R\sin\theta L(L+1)\,.\qquad(39)$$

Thus the gravitational energy contained within a sphere of radius R_1 is given by

$$\begin{aligned}P^0&\cong\frac{1}{8\pi}\int_{R\to R_1}\left\{-R\left(1-\frac{m}{R}\left[1-\tanh\left(\frac{q^2}{2mR}\right)\right]\right.\right.\\
&\left.\left.\times\left(2+\frac{3\Psi^2}{2}\right)\right)+R\left[(2+\frac{3\Psi^2}{2})\right]\right\}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&\cong m\left[1-\tanh\left(\frac{q^2}{2mR_1}\right)\right]\,,\end{aligned}\qquad(40)$$

which is the expected result.

4 Main results and discussion

In the last years a revival of non-linear electrodynamics theories is observed [70]. Non-linear electrodynamics was first proposed by Born and Infeld [71, 72] in order to obtain a finite-energy electron model. The non-linear theories appear as effective theories at different levels of string/M-theory. The Born–Infeld action arises as part of the low-energy effective action of the open superstring [73, 74].

In this paper we have studied the charged solutions in the TEGR theory, applying the most general spherically symmetric tetrad field given by (1) to the field equations (8). Exact analytic solutions are obtained by studying two cases: the case without S-term and the case with the S-term of (1).

We have obtained two exact solutions in the TEGR, (11) and (24). The associated metric of these space-times is the one of the regular charged space-time. For the tetrad field of the form (1) without the S-term, the solution can be obtained from the diagonal tetrad field [62] by applying those local Lorentz transformations that retain the form (1). Since the general expression for those local Lorentz transformations involves an arbitrary function, denoted by $\Psi(R,t)$, the solution obtained and given by (11) for the tetrad field also involves this arbitrary function and reduces to the previous solution [62] when the arbitrary function Ψ is chosen appropriately as given in (15).

For the tetrad field of the form (1) with the non-vanishing S-term, the solution (24) is derived by requiring the two conditions: the one given by (18), i.e., the condition that the second equation of (8) be satisfied, and the condition that the metric should coincide with the regular charged metric [70]. The solution involves a constant parameter η. If this constant is set equal to zero, the tetrad field (24) reduces to the matrix inverse of the solution (11) when the arbitrary function is set equal to zero, i.e., $\Psi = 0$ in Cartesian coordinates.

Maluf et al. [6, 40, 66] have derived a simple expression for the energy-momentum flux of the gravitational field. This expression is obtained on the assumption that (28) represents the energy-momentum of the gravitational field on a volume V of a three-dimensional spacelike hypersurface. They [40, 66] gave this definition for the gravitational energy-momentum in the framework of TEGR, which requires $T^{\lambda}{}_{\mu\nu}(E) = 0$ for the flat space-time. They extended this definition to the case where the flat space-time tetrad fields $E^a{}_\mu$ yield $T^a{}_{\mu\nu}(E) \neq 0$. They show [66] that in the context of the definition of the regularized gravitational energy-momentum it is not strictly necessary to stipulate asymptotic boundary conditions for tetrad fields that describe asymptotically flat space-times.

Using the definition of the torsion tensor given by (6) and applying it to the tetrad field of (11) we show that the flat space-time associated with this tetrad field has non-vanishing torsion components given by (32) and that it is adapted to an accelerated observer given by (35). However, using the regularized expression of the gravitational energy-momentum of (37) and calculating all the necessary components we finally get (40), which shows that the total energy of the tetrad field of (11), contained within a sphere of radius R_1, has a satisfactory value.

The tetrad field of (24) also suffers from the same problems and the flat space-time of (24) has the form

$$(E_a{}^{\mu}(t,r,\theta,\phi)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \dfrac{r(4r^2\eta\sin\phi + \cos\theta\cos\phi H_1)}{H_2} & \dfrac{r\sin\theta(4r^2\eta\cos\theta\cos\phi - H_1\sin\phi)}{H_2} \\ 0 & \sin\theta\sin\phi & 0 & \dfrac{r\sin\theta(4r^2\eta\cos\theta\sin\phi + H_1\cos\phi)}{H_2} \\ 0 & \cos\theta & \dfrac{-rH_1\sin\theta}{H_2} & \dfrac{-4r^3\eta\sin^2\theta}{H_2} \end{pmatrix}, \tag{41}$$

where $H_1 = (4r^4 - \eta^2)$ and $H_2 = 4r^4 + \eta^2$. It can be shown that the tetrad field $E^a{}_\mu$ of (41) has the same problems as encountered for the tetrad field $E_a{}^\mu$ of (31). Therefore, to calculate the energy associated with the tetrad field of (24) we must use the regularized expression given by (37). Calculating all the necessary components of (37) we finally obtain

$$\begin{aligned} P^0 &\cong \frac{1}{4\pi}\int_{r\to\infty}\left\{-r\left(2-\frac{m}{r}\left[1-\tanh\left(\frac{q^2}{2mr}\right)\right]\right)+2r\right\} \\ &\quad\times\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &\cong m\left[1-\tanh\left(\frac{q^2}{2mr}\right)\right]. \end{aligned} \tag{42}$$

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